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DOMINOES OVER A FREE MONOID*

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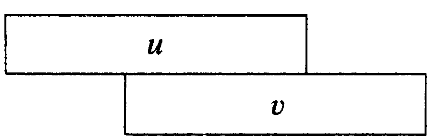
Abstract. Dominoes over a free monoid and operations on them are introduced. Related algebraic systems and their applications to decidability problems about morphisms on free monoids are studied. A new simple algorithm for testing D0L sequence equivalence is presented.

1. Introduction

Decidability and other problems concerning morphisms on free monoids have been extensively studied recently, see survey [2]. Here we introduce the notion of a domino which seems to be a very useful tool in this area. It is motivated mainly by the problem of testing the (string by string) equivalence of two morphisms on a given set of words [4]. A number of other problems reduces to such a test, for example, the D0L sequence equivalence problem [3] or equivalence problem for various types of transducers [1].

Intuitively, for any strings u, v in Σ^* we will call the construct



a domino over Σ^*, providing that the overlapping portions of u and v are identical. For a precise definition see Section 3. The strings u and v are called the components of the domino. They might not overlap at all or they might overlap completely as in the following example, which indicates the application of dominoes to testing of equality of morphisms on certain strings.

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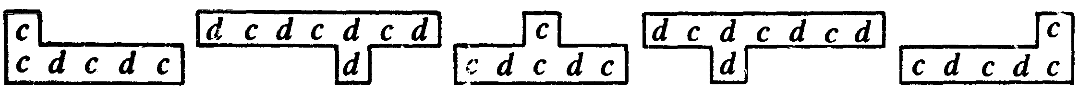
When two morphisms g and h on Σ^* are equal on a string w in Σ^*, i.e. when $g(w) = h(w)$, they might not agree on substrings of w. Consider morphisms g and h on $\{a, b\}^*$ given by

$$g(a) = c, \qquad g(b) = dcdcdcd, \qquad h(a) = cdcdc \quad \text{and} \quad h(b) = d.$$

We have $g(ababa) = h(ababa)$ but g and h differ on every substring of $ababa$. Here one of the dominoes with components $g(ababa)$ and $h(ababa)$ is

c	d	c	d	c	d	c	d	c	d	c	d	c	d	c	d	c
c	d	c	d	c	d	c	d	c	d	c	d	c	d	c	d	c

Such domino is called an equal domino. However, any decomposition of the string $ababa$ in substrings yields a decomposition of the above domino into nonequal dominoes, for example the decomposition of $ababa$ into single letters yields dominoes:



We notice that there are three distinct dominoes with the same components $g(a)$ and $h(a)$ and two distinct dominoes with components $g(b)$ and $h(b)$.

In the following two sections we formally introduce the notion of a domino, some operations on dominoes and corresponding algebraic systems. We will study their properties and also some problems concerning sets of dominoes.

In the last section we present a domino algorithm for testing D0L sequence equivalence. It is far the simplest such algorithm known to us. However, it should be stressed that we are not giving a new proof of the decidability of the D0L sequence equivalence problem. When proving that our algorithm terminates we are using the crucial property of D0L systems, namely that every two sequence equivalent D0L systems have so called bounded balance. This property has been shown for normal systems in [3] and extended to all D0L systems in [5].

2. Preliminaries

We use some elementary notions of formal language theory, we refer the reader to [7]. In the last section the theory of dominoes is applied to testing of the D0L sequence equivalence. We will remind you of the notions from [3] which we use, see also [6].

D0L system is a construct $G = (\Sigma, h, w)$ where Σ is a finite alphabet, h a morphism $\Sigma^* \to \Sigma^*$ and w in Σ^*. System G generates the language $L(G) = \{h^n(w) \mid n \geq 0\}$. Two D0L systems $G = (\Sigma, g, u)$ and $G^1 = (\Sigma, h, v)$ are *sequence equivalent* if $g^n(u) = h^n(v)$ for all $n \geq 0$.

Let g and h be morphisms on Σ^*. The *balance* of a string w in Σ^* is denoted by $B(w)$ and defined by

$$B(w) = |g(w)| - |h(w)|$$

where $|x|$ denotes the length of string x.

We say that a pair of D0L systems $G = (\Sigma, g, u)$ and $G^1 = (\Sigma, h, v)$ has *bounded balance* if there is $C > 0$ such that $|B(w)| \leq C$ for all prefixes w of all strings in $L(G)$.

3. Basics on dominoes

Let S be a set with a binary associative operation $\circ$ and A a subset of S. The semigroup, the monoid and the semigroup with zero generated by A with the operation $\circ$ are denoted by $S_\circ(A)$, $S_\circ^1(A)$ and $S_\circ^0(A)$, respectively. The corresponding semigroups with defining relations $R_1, R_2, \ldots, R_k$ are denoted by $S_\circ(A; R_1, \ldots, R_k)$, $S_\circ^1(A; R_1, \ldots, R_k)$ and $S_\circ^0(A; R_1, \ldots, R_k)$. In the case the semigroup operation is concatenation of elements we omit the operation index from the corresponding notations.

For an alphabet Σ we define the *quotient alphabet* Σ_q as the set

$$\Sigma_q = \left\{ \frac{a}{b} : a, b \in \Sigma \cup \{1\}, a \cdot b \neq 1 \right\}$$

of abstract symbols. As a convention we shall write $a/1$ as a whenever a is an element of Σ.

Let a, b and c vary over $\Sigma \cup \{1\}$ and define two generating relations E and I as follows:

$$(E) \quad a \cdot \frac{b}{c} = \frac{a}{c} \cdot b,$$

$$(I) \quad \frac{a}{a} = 1.$$

Lemma 1. *$S^1(\Sigma_q; E, I)$ is a group, with identity element* 1, *and the inverse element of a/b being b/a.*

Proof. We shall only verify that $(a/b)^{-1} = b/a$, the rest of the claim being obvious. We have

$$\frac{a}{b} \cdot \frac{b}{a} = 1 \cdot \frac{a}{b} \cdot \frac{b}{a} \overset{(E)}{=} \frac{1}{b} \cdot a \cdot 1 \cdot \frac{b}{a}$$

$$\overset{(E)}{=} \frac{1}{b} \cdot a \cdot \frac{1}{a} \cdot b \overset{(E)}{=} \frac{1}{b} \cdot \frac{a}{a} \cdot b$$

$$\overset{(I)}{=} \frac{1}{b} \cdot b \overset{(E)}{=} \frac{b}{b} \overset{(I)}{=} 1.$$

Henceforth we shall write b/a also as $(a/b)^{-1}$ and $((a/b)^{-1})^{-1}$ as a/b also when these are considered to be elements of the monoid $S^1(\Sigma_q)$. Thus a symbol $1/a$ of Σ_q will be written as a^{-1}. However, it should be remembered that this is only a conventional notation for elements of $S^1(\Sigma^q)$.

Example. A word

$$x = a \cdot b \cdot \frac{1}{a} \cdot \frac{1}{b} \cdot \frac{1}{c} \cdot \frac{b}{a}$$

may be rewritten as

$$a \cdot b \cdot (c \cdot b \cdot a)^{-1} \cdot \left(\frac{a}{b}\right)^{-1}$$

or

$$a \cdot b \cdot (a \cdot b \cdot c)^{-R}\left(\frac{a}{b}\right)^{-1},$$

where R denotes the mirror image of a word, i.e. R is the anti-isomorphism defined by $(xy)^R = y^R x^R$, and x^{-R} is a notation for $(x^{-1})^R$. The word x is reduced in $S^1(\Sigma_q)$, but in $S^1(\Sigma_q; E, I)$ we would have

$$x = c^{-1} \cdot \frac{b}{a}.$$

The rest of the paper will be devoted to special kinds of elements in $S^1(\Sigma_q)$, which we call dominoes.

A word x in $S^1(\Sigma_q)$ is called a *domino* if it has a presentation

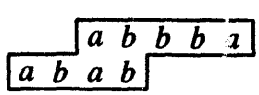
$$x = x_1 \cdot y \cdot x_2, \tag{1}$$

where x_1 and x_2 are in $S^1(\Sigma) \cup S^1(\Sigma^{-1})$ and $y = 1$ in $S^1(\Sigma_q; I)$. Here Σ^{-1} denotes the set $\{a^{-1}: a \in \Sigma\}$. The presentation (1) is canonical if x_1 is of maximal length in $S^1(\Sigma) \cup S^1(\Sigma^{-1})$ as a prefix of x. The words $x_1 = l(x)$, $y = m(x)$ and $x_2 = r(x)$ are the *left*, *middle* and *right parts* of x, respectively.

As an example let us consider a word

$$x = a^{-1} \cdot b^{-1} \cdot \frac{a}{a} \cdot \frac{b}{b} \cdot b \cdot b \cdot a. \tag{2}$$

By the definition x is a domino, since $l(x) = a^{-1}b^{-1}$ is in $S^1(\Sigma^{-1})$, $m(x) = (a/a) \cdot (b/b)$ is reducable to the identity by the generating relation I, and $r(x) = bba$ is in $S^1(\Sigma)$. Furthermore (2) is a canonical presentation of x; in fact the only presentation. The domino x may be illustrated graphically as follows:



The concept of a canonical presentation comes into use when in (1) we have $y = 1$ (in $S^1(\Sigma_q)$) and both x_1 and x_2 belong to the same semigroup $S(\Sigma)$ or $S(\Sigma^{-1})$.

Clearly, each domino has a unique canonical presentation.

Let D_Σ denote the set of dominoes in $S^1(\Sigma_q)$. The set D_Σ is not a submonoid of $S^1(\Sigma_q)$. To see this let us consider dominoes $x = a \cdot a^{-1}$ and $z = a$. Their concatenation $x \cdot z = a \cdot a^{-1} \cdot a$ is not a domino and thus D_Σ is not closed under catenation of elements. However,

Lemma 2. *D_Σ generates $S^1(\Sigma_q; E)$.*

Proof. The claim follows from the observation that by the relation E each word in $S^1(\Sigma_q)$ can be reduced to a domino.

Now we shall define some technical terms for dominoes to be used later on.

Let h_u and h_l be two morphisms from $S^1(\Sigma_q)$ into $S^1(\Sigma)$ such that if a/b is in Σ_q, then

$$h_u\left(\frac{a}{b}\right) = a \quad \text{and} \quad h_l\left(\frac{a}{b}\right) = b.$$

The words $h_u(x)$ and $h_l(x)$ are called the *upper* and the *lower components* of the domino x, respectively.

The *balance* of a domino x is an integer

$$B(x) = \max\{|l(x)|, |r(x)|\}$$

(where $|w|$ denotes the length of the word w). A domino x is said to be *fine* if it is reducable in $S^1(\Sigma_q; I)$ and x is *p-fine* if $p \cdot B(x) \leq |m(x)|$, where p is a rational number. Furthermore x is called a *B-domino,* if it is B-fine and $B(x) \leq B$.

Example. Let

$$x = a \cdot b \cdot a \cdot \frac{b}{b} \cdot \frac{b}{b} \cdot \frac{a}{a} \cdot \frac{a}{a}.$$

The upper component of x is the word

$$h_u(x) = ababbaa.$$

and the lower component of x is

$$h_l(x) = bbaa.$$

The balance of x is equal to 3 ($B(x) = \max\{3, 0\}$) and x is a fine domino, since it can be reduced by using the generating relations I.

The following lemmas are simple consequences of the definitions above.

Lemma 3. *A domino x is fine iff $m(x) \neq 1$.*

Lemma 4. *If $x = 1$ in $S^1(\Sigma_q; I)$, then $B(x) = 0$.*

If a domino x satisfies the conclusion, $B(x) = 0$, of the previous lemma we shall say that x is an *equal domino*. Thus an equal domino has empty left and right parts, and hence $x = m(x)$. By this observation we derive

Lemma 5. *Let x be an equal domino, Then x is a B-domino for each B in Q_+. As a special case of this we have that x is p-fine domino for all rational numbers p.*

Dominoes x and y are called *shiftable* with respect to each other, $x \sim y$, if $x = y$ in $S^1(\Sigma_q; E)$. The *shift* of dominoes x and y such that $x \sim y$, is the integer

$$s(x, y) = s(y, x) = \begin{cases} |l(x)| + |l(y)|, & \text{if } l(x) \in S^1(\Sigma),\ l(y) \in S^1(\Sigma^{-1}), \\ \big||l(x)| - |l(y)|\big|, & \text{if } l(x), l(y) \in S^1(\Sigma) \\ & \text{or } l(x), l(y) \in S^1(\Sigma^{-1}). \end{cases}$$

In case x and y are shiftable we also say that y is a *shift* of x.

Lemma 6. (i) *The relation $\sim$ is a congruence relation among dominoes.*
(ii) $x \sim y$ iff $h_u(x) = h_u(y)$ *and* $h_l(x) = h_l(y)$.
(iii) $s(x, y) = 0$ *iff* $x = y$.

The following lemma is frequently used in Section 5. Its proof follows closely the 'shifting argument' as given in [2].

Lemma 7. *If $x \sim y$ and $s(x, y) > 0$, then x is of the form*

$$x = \alpha_1 \beta_1 \beta^k \alpha_2, \tag{1}$$

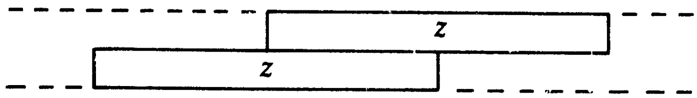
where β is an equal domino, β_1 a postfix of β, $\alpha_1 = l(x)$, $\alpha_2 = r(x)$ and $|\beta| \leq s(x, y)$. Furthermore we have

$$k \geq \left[\frac{|m(x)|}{s(x, y)}\right], \tag{2}$$

where $[c]$ denotes the greatest integer $\leq c$.

Proof. If $s(x, y) \geq |m(x)|$, then the claim follows immediately because in (2) we have only that $k \geq 1$. Assume now that $s(x, y) < |m(x)|$.

Let us consider the middle part of the domino x, i.e. the equal domino $m(x)$. Because $s(x, y) > 0$ this middle part will be also shifted by the amount $s(x, y)$ in the domino y. Thus we have the situation in y:



where $z = h_u(m(x)) = h_l(m(x))$, and hence

$$z_1 z = z z_2 \tag{3}$$

for a prefix z_1 of z and for a postfix z_2 of z each of length $s(x, y)$. Thus by (3) z can be written in the form

$$z = z_0 z_2^k$$

for a postfix z_0 of z_2 and some integer k. By choosing β to be the domino with $h_u(\beta) = h_l(\beta) = z_2$ and β_1 to be the domino with $h_u(\beta_1) = h_l(\beta_1) = z_0$, the result (1) follows. The inequality (2) is an immediate corollary of (1) and the fact that $|\beta| \leq s(x, y)$.

A domino x of the form (1) will be called *periodic*.

A domino x is called *standard* if there is no other domino y such that $x \sim y$ and either

(i) $|l(y)| < |l(x)|$ or

(ii) $|l(y)| = |l(x)|$ and $l(y) \in S^1(\Sigma)$.

Thus we require that the left part of a standard domino is of minimal length and if there are two such dominoes then we are to take the one which has the left part in $S^1(\Sigma)$. As an example let us consider a domino

$$x = a^{-1} \cdot b^{-1} \cdot \frac{b}{b} \cdot \frac{a}{a} \cdot \frac{a}{a} \cdot \frac{b}{b} \cdot \frac{b}{b}.$$

This is not a standard domino, since

$$y = b \cdot a \cdot \frac{a}{a} \cdot \frac{b}{b} \cdot \frac{b}{b} \cdot a^{-1} \cdot a^{-1} \cdot b^{-1} \cdot b^{-1}$$

fulfils the condition (ii) in the definition of a standard domino, and $x \sim y$.

In order to operate with dominoes we now introduce binary operations for dominoes. We say that a domino z is a *matching* of dominoes x and y if $z = xy$ in $S^1(\Sigma_q; E)$. We immediately derive from this definition that z is a matching of x and y if and only if $h_u(z) = h_u(x) \cdot h_u(y)$ and $h_l(z) = h_l(x) \cdot h_l(y)$.

A matching is called *standard* if it is a standard domino. From this definition we conclude the following lemma.

Lemma 8. *Each pair of dominoes (x, y) has a unique standard matching, which is denoted by $x * y$. The dominoes form a semigroup, $S_*(D_\Sigma)$, with this operation.*

We note that $S_*(D_\Sigma)$ does not contain any identity element. The empty word serves as a centre element for the semigroup: $1 * x = x * 1$ for each domino x.

However, if x is not a standard domino, then $x \neq 1 * x$. But we have the relation $x \sim 1 * x$ for all dominoes x and thus the following lemma.

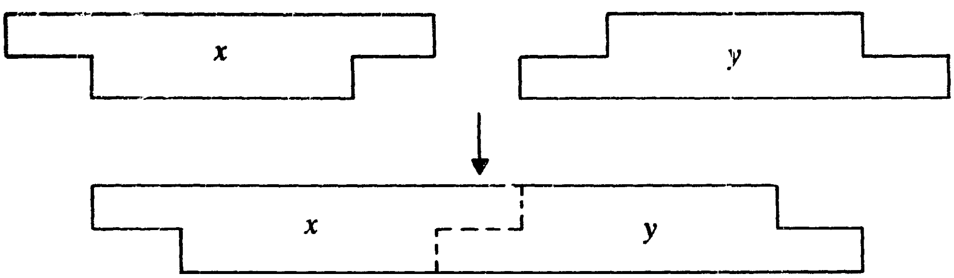
Lemma 9. *The set of standard dominoes,* DS^1_Σ, *is a monoid with the operation* $*$. *This monoid is isomorphic to the monoid* $S^1_*(D_\Sigma)/\sim$, *the factor monoid of* $S^1_*(D_\Sigma)$ *modulo the congruence* $\sim$.

The operation of standard matching does not preserve the structures of the component dominoes because of the possible shiftings made. We shall define now a new operation which preserves the structures more faithfully.

A matching z of dominoes x and y is called *faithful* if $l(z) = l(x)$ and $r(z) = r(y)$, i.e. if the left part of x and the right part of y remain the same in the matching. We immediately note that there are dominoes, which have no faithful matchings. One such example is: $x = a(a/a)$, $y = a(b/b)a$.

Lemma 10. (i) *If the faithful matching of* x *and* y *is defined, then it is unique and is denoted by* $x \times y$.

(ii) $x \times y$ *is defined iff* $r(x)^R = l(y)^{-1}$.



Proof. The lemma follows from the conditions $l(z) = l(x)$ and $r(z) = r(y)$, since these guarantee that no shifting can occur when dominoes are matched faithfully.

The operation of faithful matching is cancellable, i.e. if $x \times y = x \times z$ or $y \times x = z \times x$, then $y = z$ when these matchings are defined. The operation of standard matching does not have this property, but we have a weaker condition: $x * y = x * z$ and $y * x = z * x$ both imply that $y \sim z$. The relationship between standard and faithful matchings is given below.

Lemma 11. *For each* x *and* y *there are dominoes* x_1 *and* y_1 *such that* $x \sim x_1$, $y \sim y_1$ *and* $x * y = x_1 \times y_1$, *whenever*

$$|l(x * y)| \leq \min\{|h_u(x)|, |h_l(x)|\}$$

and

$$|r(x * y)| \leq \min\{|h_u(y)|, |h_l(y)|\}.$$

We shall now define a new operation $\odot$ in order to have faithful matchings everywhere defined. Let 0 be a new symbol and let

$$x \odot y = \begin{cases} x \times y, & \text{if defined,} \\ 0, & \text{otherwise.} \end{cases}$$

With this new operation the dominoes form a semigroup with a zero element, 0. This semigroup, denoted by $S^0_\odot(D_\Sigma)$, is not a monoid as is seen from the conditions for faithful matchings.

Lemma 12. *If* $1 \odot x \neq 0$, *then* x *has left part equal to* 1.
Furthermore if $1 \odot x \odot 1 \neq 0$, *then* x *is an equal domino.*

We shall now define and study the concept of domino equivalence, which will be an important tool in Section 5.

Dominoes x and y are *equivalent*, $x \equiv y$, if $l(x) \equiv l(y)$ and $r(x) \equiv r(y)$, i.e. if their left and right parts are equal words, respectively. The next lemma is a direct corollary to this and previous definitions.

Lemma 13. (i) *The relation* $\equiv$ *is an equivalence relation in* D_Σ.
(ii) *The relation* $\equiv$ *is a congruence relation in* $S^0_\odot(D_\Sigma)$.
(iii) $x \equiv 1$ *iff* x *is an equal domino.*
(iv) *If* z *and* y *are nonfine and* $x \equiv y$, *then* $x = y$.

4. Domino languages

We shall now study some decidability problems for families of sets of dominoes. Let $F = \{A_i : i \in J\}$ be a set of domino sets indexed over J such that each of the elements A_i is a subset of D_Σ for a fixed alphabet Σ.

The *equality problem* for F is the following decidability problem: Is it decidable if $A = \{1\}$ in $S^1(\Sigma_q; I)$ for elements A of F, i.e. is it decidable whether or not a domino set A in F consists of equal dominoes only?

The *unity problem* for F is stated as: Is it decidable if $A \cap \{1\}$ is nonempty in $S^1(\Sigma_q; I)$ for elements A of F? This problem may be restated in the form: Is it decidable whether or not a domino set A in F contains an equal domino?

The *finiteness problem* for F asks if it is decidable whether or not an element A of F is finite?

Our first result deals with families F the elements of which are finite sets of dominoes. Since the claim of the theorem is obvious we shall omit the proof.

Theorem 1. *The equality, unity and finiteness problems are decidable for families* F *consisting of finite sets of dominoes.*

A more interesting case arises when we allow the elements of F to be infinite. In the next theorem we let the domino sets vary through the semigroups $S_*(A)$ and $S^0_\odot(A)$, where A is a finite set of generators.

Theorem 2. *Let $F = \{A_i : i \in J\}$ be the family of finite domino sets. Then the following hold true:*

(i) *The equality and finiteness problems are decidable for*

$$F_1 = \{S^0_\odot(A_i) : i \in J\} \quad \text{and} \quad F_2 = \{S_*(A_i) : i \in J\}.$$

(ii) *The unity problem is undecidable for F_2 but decidable for F_1.*

Proof. (i) The semigroups $S^0_\odot(A)$ and $S_*(A)$ consist of equal dominoes if and only if the set A consists of equal dominoes. The latter case is clearly decidable since A is a finite set. Thus the equality problem is decidable for both F_1 and F_2.

For the finiteness problem let us first consider the family F_2. If $A = \{1\}$, then we shall have also $S_*(A) = \{1\}$. On the other hand suppose A has a nonempty word x. In this case $S_*(A)$ contains the dominoes

$$x_k = x * x * \cdots * x,$$

where the product has k factors x, for each $k \geq 1$. By the definition of standard matching the length sequence $\{|x_k|\}_{k=1}^{\infty}$ must tend to infinity. Hence $S_*(A)$ is infinite if and only if A is nonempty and different from $\{1\}$.

For the finiteness conditions for $S^0_\odot(A)$ let us assume that $\# A = k$, i.e. the cardinality of A equals k. If $k = 0$, then $S^0_\odot(A)$ is finite. Suppose that $k \geq 1$, and that $S^0_\odot(A)$ contains a word x such that

$$x = x_1 \times x_2 \times \cdots \times x_{k+1}, \quad x_i \in A. \tag{1}$$

Since there are more factors in (1) than the cardinality of A two of these factors must be equal, say x_i and x_j. Then

$$x = x_1 \times \cdots \times x_i \times (x_{i+1} \times \cdots \times x_j) \times x_{j+1} \times \cdots \times x_{k+1}$$

and by the definition of faithful matching we have that $l(x_{i+1})^R = r(x_i)^{-1}$. Now $x_i = x_j$ and thus the subword $x_{i+1} \times \cdots \times x_j$ may be repeated indefinitely, i.e. the words

$$x_{i+1} \times \cdots \times x_j,\ x_{i+1} \times \cdots \times x_j \times x_{i+1} \times \cdots \times x_j,\ \cdots$$

are all different from 0. Thus the condition (1) implies that $S^0_\odot(A)$ is infinite. On the other hand if there are no words x satisfying the condition (1), then $S^0_\odot(A)$ is finite. Therefore, the finiteness problem is decidable for F_1, too.

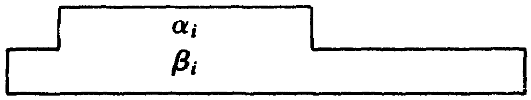
(ii) The proof of the decidability of the unity problem for F_1 follows closely the above argumentation for finiteness and is omitted here.

In order to prove that the unity problem is undecidable for F_2 we shall consider instances of the Post correspondence problem, PCP in short. Let

$$(\alpha_1, \alpha_2, \ldots, \alpha_k; \beta_1, \beta_2, \ldots, \beta_k) \tag{2}$$

be an instance of the PCP. Define a finite set A of dominoes by setting an element

$$x_i = \alpha_i * \beta_i^{-R},$$



into A for each $i = 1, 2, \ldots, k$. By this construction it immediately follows that $S_*(A)$ contains an equal domino $x = x_{i_1}x_{i_2}\ldots x_{i_n}$ if and only if $i_1 i_2 \ldots i_n$ is a solution to the problem (2). Because the PCP is an undecidable problem so must the unity problem for F_2.

From the proof of the previous theorem we already see that the semigroups $S^0_{\odot}(A)$ possess some regularity properties. We shall formalize this in the following way.

Theorem 3. *Let A be a finite set of dominoes and ρ an isomorphism from an alphabet Δ onto the set A. Then the set*

$$B_\rho = \{w \mid w \in S(\Delta), \rho(W) \in S^0_{\odot}(A) \setminus \{0\}\}$$

is a regular set.

Proof. Let $A = \{x_1, x_2, \ldots, x_k\}$ and $\Delta = \{b_1, b_2, \ldots, b_k\}$. We shall design a right linear grammar for B_ρ. This grammar has starting letter S, nonterminals $Y_1, Y_2, \ldots, Y_k$, and productions:

$$S \to Y_i \qquad (i = 1, 2, \ldots, n).$$

$$Y_i \to b_i Y_j$$

for each i and j in $\{1, 2, \ldots, n\}$ such that x_i and x_j are faithfully matchable, and

$$Y_i \to b_i \quad (i = 1, 2, \ldots, n).$$

From this construction the claim follows immediately.

A direct corollary to this result states that the upper and lower components of dominoes in $S^0_{\odot}(A)$ have regularity properties.

Corollary 4. *Let A be a finite set of dominoes. Then $h_u(S^0_{\odot}(A))$ and $h_l(S^0_{\odot}(A))$ are regular sets.*

The above considerations can be generalized to wider classes of dominoes. Let Δ be an alphabet and ϕ a morphism from $S(\Delta)$ into $S_*(D_\Sigma)$. For a subset L of $S(\Delta)$ we define the set $\phi(L)$ to be a *domino language* of L defined by ϕ. Similarly a *faithful domino language* of L defined by a morphism $\psi: S(\Delta) \to S^0_{\odot}(D_\Sigma)$ is the set $\psi(L)$ of dominoes.

Theorem 5. *If A is a subset of a finitely generated semigroup $S_*(A_1)$, then A is a domino language (of some L).*

Proof. First of all we note that $S_*(A_1)$ is a domino language of $S(\Delta)$ defined by an isomorphism $\phi:\Delta\to A_1$. Now A is a domino language of L defined by ϕ, where L is a set

$$L=\{b_1b_2\cdots b_n \mid b_i\in\Delta,\ \phi(b_1b_2\cdots b_n)\in A\}.$$

In quite the same way we can prove

Theorem 6. *If A is a subset of a finitely generated semigroup $S^0_{\odot}(A_1)$, then A is a faithful domino language (of some L).*

By these two theorems we may consider domino languages and faithful domino languages instead of sets of dominoes.

Next we shall reduce the domino problems (equality and unity) to problems concerning ordinary formal languages.

A *morphic equality problem* for a language L is the problem: Given two morphisms h and g do they coincide on L, i.e. does $h(w)=g(w)$ hold true for each w in L?

Theorem 7. *The morphic equality problem for L is equivalent to the equality problem for domino languages of L.*

Proof. Let L be a subset of $S^1(\Delta)$ and h, g two morphisms from $S^1(\Delta)$ into $S^1(\Sigma)$. Define a morphism $I_q:S^1(\Sigma)$ into $S^1(\Sigma^q)$ by setting $I_q(a)=a/1$ and let

$$\phi(w)=I_qh(w)*(I_qg(w))^{-R}$$

be a morphism from $S(\Delta)$ into $S_*(D_\Sigma)$. Then by the definition of ϕ we have that $h(w)=g(w)$ if and only if $\phi(w)$ is an equal domino.

On the other hand let ϕ be given for L. We shall now define two morphisms h and g by

$$h(w)=h_u\phi(w) \quad \text{and} \quad g(w)=h_l\phi(w).$$

Now $h(w)=g(w)$ if and only if $\phi(w)$ is an equal domino.

5. D0Ls and dominoes

In this section we apply dominoes to the D0L sequence equivalence problem in order to obtain for it a simple new algorithm.

Given two morphisms h and g on an alphabet Σ and a word w_0 in $S^1(\Sigma)$, we ask whether $h^n(w_0)=g^n(w_0)$ for all integers $n=0,1,2,\ldots$. This problem can be restated using the following lemma.

Lemma 14. *$h^n(w_0)=g^n(w_0)$ for all n if and only if $h^{n+1}(w_0)=gh^n(w_0)$ for all n.*

Proof. Obvious.

Hence, by Theorem 7 the D0L sequence equivalence problem is equivalent to the equality problem for a domino language of $L=\{h^n(w_0): n\geq 0\}$ defined by a morphism ϕ such that

$$\phi(w)=I_q h(w)*(I_q g(w))^{-R}.$$

We shall now fix the morphisms h and g as well as the starting string w_0. We begin our considerations with some definitions.

Let x_1 be a given domino. Then x_1 will result in a sequence of new dominoes $x_2, x_3, \dots$ defined by iterative using of the morphisms h and g. Suppose that $h_u(x_1)=w_1$ and $h_l(x_1)=w_2$. At the first step x_1 results a domino

$$x_2=h(w_1)*(g(w_1))^{-R}.$$

In general we have

$$x_{i+1}=h(h_u(x_i))*(g(h_u(x_i))^{-R},$$

which can be written as

$$x_{i+1}=\phi h^i(w_1)=\phi h^i h_u(x_1).$$

We shall say that a domino x is *repetitive modulo t* ($t>1$) if $x\equiv y$ for some y such that $y\sim\phi h^t h_u(x)$. Furthermore x is a *(B,t)-domino* if it is a B-domino and is repetitive modulo t.

In the following lemmas we study the properties of (B,t)-dominoes in the framework of the system (h, g, w_0).

Lemma 15. *Suppose x_1 and x_2 are (B,t)-dominoes such that $x_1\sim x_2$. Then either $x_1=x_2$ or x is repetitive modulo t whenever $x\sim x_1$.*

Proof. Assume that $x_1\neq x_2$ in which case $s(x_1,x_2)\neq 0$ and $s(x_1,x_2)\leq 2\cdot B$. Let $x_{i,t}$ be a domino such that $x_{i,t}\sim\phi h^t h_u(x_i)$ for $i=1,2$. Since $x_1\sim x_2$ we have also that $x_{1,t}\sim x_{2,t}$. Moreover, $x_i\equiv x_{i,t}$ implies that $s(x_{1,t},x_{2,t})=s(x_1,x_2)$, where $i=1,2$.

By Lemma 6 the following hold

$$x_1=\alpha_1\beta^k\beta_1\alpha_2, \tag{1}$$

$$x_{1,t}=\alpha_1\beta^r\beta_2\alpha_2 \tag{2}$$

for some equal domino β, β_1 and β_2 being prefixes of β, $k\geq[\frac{1}{2}|m(x_1)|/B(x_1)]$, and $\alpha_1=l(x_1)$, $\alpha_2=r(x_1)$. Here the period β may be assumed to be of minimal length.

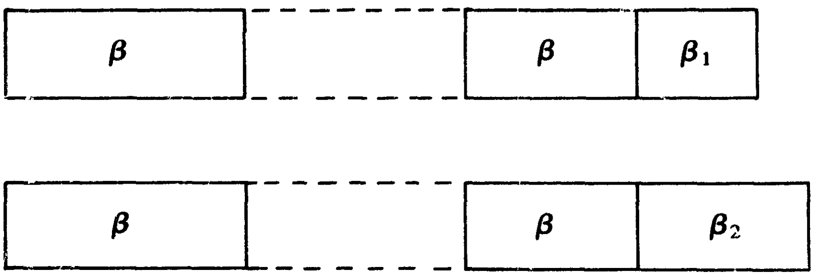
Claim. $\beta_1=\beta_2$.

Assume the contrary, i.e. $\beta_1\neq\beta_2$.

The dominoes x_i and $x_{i,t}$ $(i=1,2)$ can be written as

$$x_i \sim \alpha_1 * z_i * \alpha_2 \quad \text{and} \quad x_{i,t} \sim \alpha_1 * z_{i,t} * \alpha_2.$$

The equivalences $x_i \equiv x_{i,t}$ imply that also $z_i \equiv z_{i,t}$ for $i=1,2$. Now the dominoes z_1 and $z_{1,t}$ look like



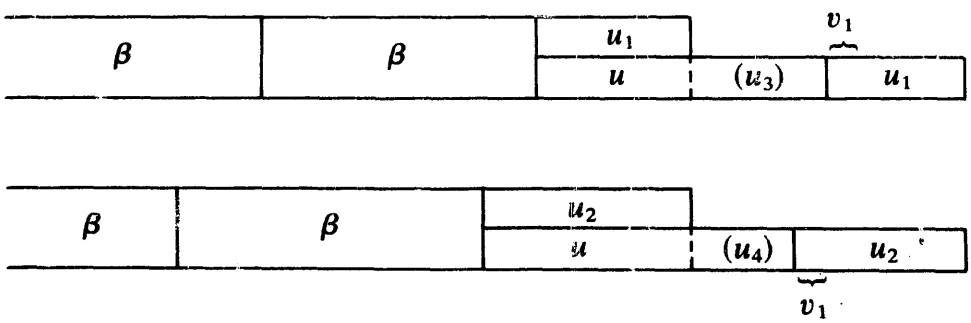
The dominoes z_2 and $z_{2,t}$ are equivalent and thus their right parts are equal to each other. Because of $x_1 \sim x_2$ and $x_{1,t} \sim x_{2,t}$ this may happen only if the upper component of x_1 is shifted some β blocks to the right or to the left. These two shifting directions are clearly symmetrical when we consider the dominoes z_i and $z_{i,t}$ and thus the same proof applies to both of them.

This shifting and the equivalence $z_2 \equiv z_{2,t}$ imply that

$$r(z_2) = u_3 u^v u_1 = u_4 u^v u_2 = r(z_{2,t}), \tag{3}$$

where $u = h_u(\beta)$, $v \geq 0$, $u_i = h_u(\beta_i)$ for $i=1,2$, and

$$u = u_1 u_3 = u_2 u_4. \tag{4}$$



By assumption $\beta_1 \neq \beta_2$ we have that $u_1 \neq u_2$ and thus also $u_3 \neq u_4$. Let us suppose that $|u_2| > |u_1|$. The reverse case is clearly symmetrical. We have

$$u_2 = v_1 u_1, \qquad u_3 = u_4 v_1 \tag{5}$$

for some nonempty word v_1. In all we obtain from (4) and (5) that

$$u = u_1 u_4 v_1 = v_1 u_1 u_4. \tag{6}$$

Here $u_1 u_4$ and v_1 are nonempty words which commute in u. Thus u is a periodic word of the form v^k form some $k \geq 2$. Hence the domino $\beta = u * u^{-R}$ would be periodic contradicting our assumption of the minimality of $|\beta|$ in (1). The proof of the claim is completed by this contradiction.

The claim now implies that x_1 and $x_{1,t}$ are always shiftable by the same amount because in the equations (1) and (2) we have that $\beta_1=\beta_2$. The results of these shiftings remain equivalent, and thus the lemma follows.

Lemma 16. *Let $x_1\times y_1\times z_1$ be a faithful matching of (B,t)-dominoes x_1, y_1 and z_1. Suppose $x_2\times y_2\times z_2$ is a matching of (B,t)-dominoes $x_2\sim x_1$ and $z_2\sim z_1$, and a B-domino $y_2\sim y_1$. Then y_2 is a (B,t)-domino.*

Proof. Suppose that $y_2\neq y_1$ in which case also $x_1\neq x_2$ and $z_1\neq z_2$. Now the dominoes $x_1\times y_1\times z_1$ and $x_2\times y_2\times z_2$ are shiftable with respect to each other and thus

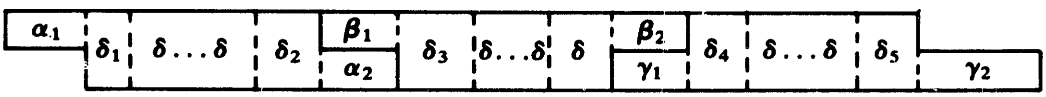
$$x_1\times y_1\times z_1=\alpha_1\delta_1\delta^{k}\delta_5\gamma_2, \tag{1}$$

$$x_1=\alpha_1\delta_1\delta^{k_1}\delta_2\alpha_2, \tag{2}$$

$$y_1=\beta_1\delta_3\delta^{k_2}\beta_2, \tag{3}$$

$$z_1=\gamma_1\delta_4\delta^{k_3}\delta_5\gamma_2, \tag{4}$$

where δ is a minimal period and $\delta_2\times(\alpha_2*\beta_1)\times\delta_3=\delta^{v_1}$, $(\beta_2*\gamma_1)\times\delta_4=\delta^{v_2}$ for some integers v_1 and v_2. Note that $\alpha_2=\beta_1^{-R}$ and $\beta_2=\gamma_1^{-R}$.



The dominoes x_1 and x_2 are both (B,t)-dominoes such that $x_1\sim x_2$. By the proof of Lemma 15 we obtain that

$$x_{1,t}=\alpha_1\delta_1\delta^{r_1}\delta_2\alpha_2, \tag{5}$$

for some integer r_1 (Here $x_1\equiv x_{1,t}$ modulo t). The same argument applies to $z_{1,t}\equiv z_1$

$$z_{1,t}=\gamma_1\delta_4\delta^{r_3}\delta_5\gamma_2, \tag{6}$$

for some integer r_3.

Moreover $y_1\equiv y_{1,t}$ and the domino $y_{1,t}$ is now sandwiched between $x_{1,t}$ and $z_{1,t}$ in the domino $x_{1,t}\times y_{1,t}\times z_{1,t}$. This condition with the periodicity property in (1) implies that

$$y_{1,t}=\beta_1\delta_3\delta^{r_2}\beta_2 \tag{7}$$

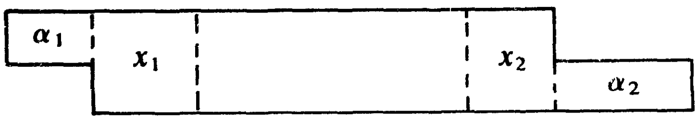
for some integer r_2. By (3) and (7) the result follows.

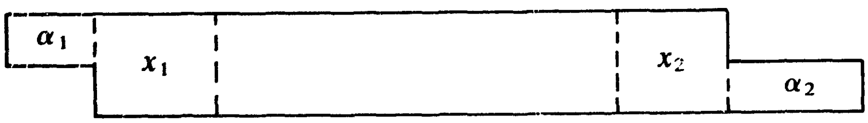
We shall call a B-domino x *unique* (with respect to B and t) if x has exactly one shifting y such that y is a (B,t)-domino. By Lemma 15 if x is not unique, then it has either no shiftings, which are (B,t)-dominoes or all its shiftings which are B-dominoes are also (B,t)-dominoes.

With this terminology Lemma 16 can be reformulated as follows:

Lemma 17. *Let $x = x_1 \times y_1 \times z_1$ be a faithful matching of (B, t)-dominoes x_1, y_1 and z_1, where x_1 and z_1 are nonunique. If x has a proper shift $x_2 \times y_2 \times z_2$, where x_2, y_2 and z_2 are B-dominoes, then y_1 is nonunique.*

In order to simplify the conditions for the equality problem we shall now introduce a stronger version of the equivalence relation for dominoes. The dominoes x and y are *B-equivalent*, $x \equiv_B y$ if they are equivalent and $m(x)$, $m(y)$ have common prefix and postfix of length B.





Clearly, B-equivalence implies equivalence. Analogously, we call a domino x *strongly repetitive modulo* $t \geq 1$ if $x \equiv_B y$ for some y such that $y \sim \phi h^t h_u(x)$. Moreover x is a *strong* (B, t)*-domino* if it is a B-domino and is strongly repetitive modulo t.

We now prove that a faithful matching of unique strong dominoes is unique.

Lemma 18. *Let $x_1 \times y_1$ be a faithful matching of unique strong (B, t)-dominoes x_1 and y_1. If $y = x_2 \times y_2$ is a (B, t)-domino, where $x_2 \sim x_1$ and $y_2 \sim y_1$ are B-dominoes, then $x = y$.*

Proof. Suppose that $x \neq y$. Let x_t, y_t, $x_{1,t}$ and $y_{1,t}$ be the corresponding repetitions mod t of x, y, x_1 and y_1. We shall write $y_t = x_{2,t} \times y_{2,t}$, where $x_{1,t} \sim x_{2,t}$ and $y_{1,t} \sim y_{2,t}$.

From the assumption $x \neq y$ and from the uniqueness of x_1 and y_1 we conclude that $x_2 \not\equiv x_{2,t}$ and $y_2 \not\equiv y_{2,t}$. This means that $r(x_2) \neq r(x_{2,t})$. However, $l(x_2) = l(x_{2,t})$ since $y \equiv y_t$ and thus $|r(x_2)| = |r(x_{2,t})|$.

The dominoes x and y are shiftable and hence periodic with a minimal period β. This holds also for x_t and y_t since the shift must be the same for both of these pairs and because of the equivalences $x \equiv x_t$ and $y \equiv y_t$ the minimal period is β for all of these four dominoes.

Now x_1 will be of the form

$$x_1 = \alpha_1 \beta_1 \beta^k \alpha_2$$

and $x_{1,t}$ of the form

$$x_{1,t} = \alpha_1 \delta_1 \delta^k \alpha_2,$$

for some δ, which is a cyclic conjugate of β, i.e. $\beta=\beta_2\beta_3$ and $\delta=\beta_3\beta_2$ for some β_2 and β_3. But since we assumed that x_1 is strongly repetitive, we must have that $\beta=\delta$. By the same reason we have also that $\beta_1=\delta_1$. These conditions would imply that x_2 is a (B,t)-domino by the proof of Lemma 15. This contradicts the uniqueness assumption for x_1. Hence $x_2=x_1$ and in all $x=y$.

In the next lemma we consider sequences of matchable dominoes.

Lemma 19. *Let $x=x_0\times x_1\times\cdots\times x_k$ be a faithful matching of B-dominoes $x_0,x_1,\ldots,x_k$ such that x_0 is a (B,t)-domino and*

> *each pair (x_i,x_{i+1}) has a matching $z_i=y_i\times u_i$, where y_i and u_i are strong (B,t)-dominoes and $y_i\sim x_i$, $u_i\sim x_{i+1}$ for each $i=0,1,\ldots,k-1$.* (1)

Then either all of the dominoes x_i are (B,t)-dominoes or there is an integer j such that $x_0,x_1,\ldots,x_{j-1}$ are (B,t)-dominoes and $x_j,\ldots,x_k$ are unique, but not (B,t)-dominoes.

Proof. Assume that not all of the dominoes x_i, $i=1,2,\ldots,k$, are (B,t)-dominoes and let x_j be the first domino in x which is not a (B,t)-domino. Then $j\geq 1$ and x_j must be unique by Lemma 15. The domino x_{j-1} is nonunique because it is a (B,t)-domino and can be shifted to another (B,t)-domino by Condition (1) and by the fact that x_j is a unique domino in a wrong position, i.e. $x_j\neq u_{j-1}$.

Suppose now that x_{j+r} is a (B,t)-domino for some minimal $r\geq 1$. Then x_{j+r} is nonunique by the same arguments as for x_{j-1}. Thus the dominoes $x_j,\ldots,x_{j+r-1}$ are all unique but not (B,t)-dominoes. Let

$$y=y_j\times y_{j+1}\times\cdots\times y_{j+r-1}$$

be a domino obtained from Condition (1). Here y_{j+i} is a unique strong (B,t)-domino for each $i=0,1,\ldots,r-1$. By Condition (1) also x_{j-1} and x_{j+r} may be shiftably matched to y yielding a domino

$$y_{j-1}\times y_j\times\cdots\times y_{j+r-1}\times u_{j+r-1} \tag{2}$$

all the factors of which are strong (B,t)-dominoes.

Applying Lemma 18 to the domino y we obtain that y is a unique domino itself. But from (2) we have a domino $z=y_{j-1}\times y\times u_{j+r-1}$, where y_{j-1} and u_{j+r-1} are nonunique. Furthermore, the domino z is properly shiftable to the domino $x_{j-1}\times x_j\times\cdots\times x_{j+r}$ and thus y must be nonunique by Lemma 17. This contradicts our previous result for the uniqueness of y. In all the counter assumption fails and hence the claim follows.

Now, we shall construct the necessary and sufficient conditions for a solution to the domino equality problem.

Let us denote the D0L language $\{h^n(w_0): n \geq 0\}$ by L. We define the set of adjacent symbols in L as

$$\mathrm{Init}(L) = \{w: w_1 w w_2 \in L \text{ for some } w_1 \text{ and } w_2\}.$$

Furthermore, two words w_1 and w_2 are said to be *adjacent* in L if $w_1 w_2$ is in $\mathrm{Init}(L)$. Dominoes $x = \phi h^n(w_1)$ and $y = \phi h^n(w_2)$ are *adjacent* in $\phi(L)$ if w_1 and w_2 are adjacent in L.

Let A be a set of dominoes and B, n positive integers. The set A is called a *base* w.r.t. B and n (i.e., with respect to B and n), if A consists of dominoes x such that either (1)

$$x = \phi h^n(a) \quad \text{and} \quad |x| \geq B^2$$

for a letter a in Σ, or

$$x = \phi h^n(w) \quad \text{and} \quad B^2 \leq |x| \leq 2 \cdot B^2$$

for a word w in $\mathrm{Init}(L)$, or (2)

$$x = x_1 \times x_2 \times x_3,$$

where x_2 is from the case (1) and $x_i \sim \phi h^n(w_i)$, $|x_i| < B^2$ such that w_i is in $\mathrm{Init}(L)$ for $i = 1, 3$.

Thus a base (w.r.t. B and n) consists of dominoes of the form $\phi h^n(w)$ which are sufficiently large with respect to B. A base is always a finite set.

We shall now state the necessary and sufficient conditions for a domino language $\phi(L)$ to be a set of equal dominoes.

Condition 1. There are positive integers B, n and t, and a *base* A of $\phi(L)$ w.r.t. B and n such that

(1.1) If elements x and y of A are adjacent in $\phi(L)$, then they have faithfully matchable (B, t)-dominoes x_1 and y_1 as shifts. Furthermore, if $x_{1,t} \sim \phi h^t(x)$, $y_{1,t} \sim \phi h^t(y)$ and $x_{1,t} \equiv x_1$, $y_{1,t} \equiv y$, then the factors of $x_{1,t}$ and $y_{1,t}$ from A are B-dominoes.

(1.2) If x is in A and begins (ends, resp.) a word in $\phi(L)$ then x has a (B, t)-domino x_1 as a shift such that $|l(x_1)| = 0$ ($|r(x_1)| = 0$, resp.).

(1.3) $\phi h^j(w_0)$ is an equal domino for $n \leq j \leq n + t$, the factors (in A) of which are B-dominoes.

The next lemma shows that this condition is sufficient to guarantee that $\phi h^j(w_0)$ is equal for each $j \geq 0$.

Lemma 20. *Condition* 1 *implies that* $\phi h^j(w_0)$ *is an equal domino for all* $j \geq 0$.

Proof. From Condition 1.3 we know that $\phi h^j(w_0)$ is an equal domino for $j \leq n + t$.

Let us consider the equal domino $\phi h^{n+1}(w_0)$. We have that

$$\phi h^{n+1}(w_0) = x_0 \times x_1 \times \cdots \times x_r \tag{1}$$

for some B-dominoes x_i in A by assumption. Moreover the domino x_0 is a (B, t)-domino by Condition 1.2. Hence by Lemma 19 either all the dominoes x_i are (B, t)-dominoes (for $i=0, 1, \ldots, r$) or there is an integer j such that $x_j, \ldots, x_r$ are unique dominoes, which are not (B, t)-dominoes. The second case does not hold true since by Condition 1.2 the domino x_r has a shift, which is a (B, t)-domino with right part equal to the empty word, but x_r itself has $|r(x_r)| = 0$ and thus it must be a (B, t)-domino.

From this reasoning we conclude that the dominoes $x_0, x_1, \ldots, x_r$ are (B, t)-dominoes. This implies that $\phi h^{n+t+1}(w_0) \equiv \phi h^{n+1}(w_0)$ and thus $\phi h^{n+t+1}(w_0)$ is an equal domino. By Condition 1.1. the equal domino $\phi h^{n+t+1}(w_0)$ is expressible as a faithful matching of B-dominoes from A.

By proceeding inductively we obtain the result of the lemma.

The necessity of Condition 1 follows from the crucial property of D0L systems, namely that every two equivalent D0L systems have a bounded balance. This property has been shown in [3] for 'normal' D0L systems as essential step in proving the decidability of the D0L sequence equivalence problem. It has been extended to all D0L systems in [5].

Lemma 21. *If $\phi h^j(w_0)$ is an equal domino for each $j \geq 0$, then Condition 1 holds.*

Proof. Let us suppose that $L = \{h^j(w_0): j \geq 0\}$ is infinite. Otherwise the claim is trivial.

The dominoes $\phi h^j(w_0)$ are equal dominoes for all $j \geq 0$ if and only if $h^{j+1}(w_0) = gh^j(w_0)$ holds for all $j \geq 0$. In particular this implies that if $\phi(L) = \{\phi h^j(w_0): j \geq 0\}$ is a set of equal dominoes then h and g have a bounded balance, i.e.

$$||h(w)| - |g(w)|| \leq B$$

for all words w, which are prefixes of words in L.

Let a be a letter in Σ and consider a word

$$h^j(w_0) = w_1 a w_2, \tag{1}$$

where $j \geq 0$ and w_1, w_2 are words in Init(L). Applying the morphism h repetitively we obtain

$$h^{j+k}(w_0) = h^k(w_1) h^k(a) h^k(w_2)$$

and

$$\phi h^{j+k}(w_0) = u_k \times z_k \times v_k$$

for $k \geq 0$, where $u_k = \phi h^k(w_1)$, $z_k \sim \phi h^k(a)$ and $v_k \sim \phi h^k(w_2)$ are all dominoes with balance at most B.

The infinite sequence $\{z_i\}$ *produced* by an occurrence of a, in (1), contains only dominoes which have balance bounded by B. Thus in this sequence there are only finitely many equivalence classes with respect to the relation of equivalence of dominoes. Furthermore, each of the dominoes z_{i+1}, $i = 1, 2, \ldots$, is produced from z_i by using the morphisms h and g. This means that the equivalent dominoes in the sequence $\{z_i\}$ occur periodically, i.e. there are integers s and r such that $z_i \equiv z_{i+r}$ whenever $i \geq s$.

Different occurrences of a may produce different sequences, but since the elements of each of these sequences have balance bounded by B, there are only a finite number of differnt ones. Thus we may select integers n_a and t_a such that every sequence $\{f_i\}$ produced by a is periodic, that is

$$f_i \equiv f_{i+t_a}$$

whenever $i \geq n_a$.

Let $n_0 = \max\{n_a : a \in \Sigma\}$ and $t_0 = [t_a : a \in \Sigma]$, the least common multiple of the integers t_a, $a \in \Sigma$. By this choice if $\{p_i\}$ is a sequence produced by some occurrence of a letter then

$$p_i \equiv p_{i+t_0} \tag{2}$$

for all $i \geq n_0$.

Let $n_1 \geq n_0$ be an integer such that either $|h^{n_1+i}(a)| \geq B^2$ for each $i \geq 0$ or the set $\{h^j(a): j \geq 0\}$ is finite, where a runs through the letters of the alphabet Σ. Furthermore, let A_0 be a base w.r.t. B and n_1 for $\phi(L)$.

The elements of A_0 are all B-dominoes and each of them has a shift, which is a (B, t_0)-domino by the above arguments. Now we shall repeat the above considerations for elements of A_0 to obtain the result.

Claim. There are integers n and t such that if an occurrence of an element in A_0 produces an infinite sequence $\{p_i\}$, then

$$p_i \equiv_B p_{i+t},$$

whenever $i \geq n$.

Since the proof of this fact follows closely the proof of (2) we omit it here.

The lemma follows immediately from this claim when we select A as a base w.r.t. B and n for $\phi(L)$.

We remind here that L is a D0L language of the form $\{h^n(w_0): n \geq 0\}$ and that the morphism ϕ is defined using the given morphisms h and g as indicated in the beginning of the chapter. For clarity we shall write also ϕ as $\phi_{h,g}$ in order to specify the morphisms h and g which define ϕ.

From the previous two lemmas we deduce

Theorem 8. *Condition* 1 *holds for the domino language* $\phi_{h,g}(L)$ *if and only if* $\phi_{h,g}(L)$ *is a set of equal dominoes.*

A direct corollary to this theorem states

Corollary 9. *Let $L_1 = \{h^n(w_0): n \geq 0\}$ and $L_2 = \{g_n(w_0): n \geq 0\}$ be two D0L languages. Condition 1 holds for the domino language $\phi_{h,g}(L_1)$ if and only if $g^n(w_0) = h^n(w_0)$ for all $n \geq 0$.*

Thus Condition 1 provides a necessary and sufficient condition for two D0L sequences to be equal.

By Corollary 9 an algorithm for testing Condition 1 will be also an algorithm for the D0L sequence equivalence problem. We shall now proceed to give such an algorithm for Condition 1.

Let $G_1 = (\Sigma, h, w_0)$ and $G_2 = (\Sigma, g, w_0)$ be two D0L systems and let

$$\Sigma = \{a_1, a_2, \ldots, a_r\}.$$

We shall denote by Σ' the set of all letters of Σ such that the morphism h is growing on these, i.e.

$$\Sigma' = \{a: a \in \Sigma \text{ and } \{|h^n(a)|: n = 0, 1, \ldots\} \text{ is an infinite set}\}.$$

The algorithm advances in stages n, the initial stage being $n = 1$.

Algorithm for testing Condition 1.

Stage n.

(i) Let

$$B_n = \max\{B(x): x \text{ is a factor of } \phi h^n(w_0) \text{ of the form } x \sim \phi h^n(a_i)\}.$$

(ii) Let B be the maximal integer such that $B \geq B_n$ and $|\phi h^n(a_i)| \geq B^2$ for each a_i in Σ'. If no such B exists, i.e. if $|\phi h^n(a_i)| < B_n^2$, then go to stage $n+1$.

(iii) Construct the base A w.r.t. B and n.

(iv) Evolve the sequences B_{n+j}, $\phi h^{n+j}(w_0)$ and $\phi h^{n+j}(u)$ from $j = 1$ onwards, for each element u of A, until either

(iv.1) $\phi h^{n+j}(w_0)$ is nonequal in which case Condition 1 is not true, or

(iv.2) $B_{n+j} > B$ in which case the procedure will continue to stage $n+j$, or

(iv.3) all dominoes u_1 and u_2 from A which are adjacent in $\phi(L)$ have faithfully matchable strong (B, j)-dominoes as shifts. In this case the algorithm stops and Condition 1 is true.

We are now to prove that the above algorithm is an effective method for testing Condition 1.

Theorem 10. *The above algorithm is a test algorithm for Condition 1.*

Proof. Clearly parts (i) and (ii) are effective. Part (iii) is effective since a base is always a finite set which can be constructed by first constructing the finite set

$$\mathrm{Init}_M(L) = \{w\colon w \in L \text{ and } |w| \leq M\},$$

where

$$M = \max\{|h^n(a_i)| \; a_i \text{ is in } \Sigma\}.$$

Ir remains to prove that the part (iv) of the algorithm will be finitely processed during each stage n and that there is a stage n which ends up to cases (iv.1) or (iv.2).

The first of these claims follows immediately from the proof of Lemma 21 since otherwise the condition (iv.1) implies that the dominoes $\phi h^{n+j}(w_0)$ are equal for all $j \geq 0$, the condition (iv.2) implies that the factors of each $\phi h^{n+j}(w_0)$ have bounded balance and these two results would contradict the part (iv.3) and the claim presented in the proof of Lemma 21.

If Condition 1 does not hold, then there is an integer n such that $\phi h^n(w_0)$ is nonequal, in which case the algorithm would stop at stage n and would reveal this nonequal domino in part (iv.1).

On the other hand if Condition 1 holds true for $\phi(L)$, then eventually a correct base is found for $\phi(L)$ w.r.t. some B and n. This happens at stage n and the algorithm will stop in part (iv.3) when an integer $j = t$ is reached which fulfils the requirements of Condition 1.

The above algorithm serves as an effective procedure for testing D0L sequence equivalence problem.

Theorem 8 has also another direct corollary.

Corollary 11. *Let h and g be two morphisms and* $L = \{h^n(w_0)\colon n \geq 0\}$. *If the domino language* $\phi_{h,g}(L)$ *is a set of equal dominoes, then it is included in a finitely generated semigroup* $S^0_{\odot}(A)$, for some *set* A of dominoes.

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